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ON A BASIC CONSIDERATION OF THE WARP MODEL OF HOUGH TRANSFORM

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ABSTRACT

The Warp Model of the Hough transform is proposed to provide the extended Hough transform functions(EHT). The Warp Model is a skewed parameter space (μ,ξ), *which is homeomorphic to the original* (ρ,θ) *parameter space. It is important that to introduce the skewness of the parameter space is to define the angular and positional sensibility to detect lines.*
In this paper, the Hough transform function is extended at first by introducing three functional conditions to ensure the homeomorphic relation between (μ,ξ) *and* (ρ,θ) *parameter spaces. After proving these conditions to be feasible, a procedure to derive the transform function is presented by using the Warp Model so that the given angular and posional sensibility could be realized.*

1. INTRODUCTION

The basic function of Hough transform defined by eq.(1) should not be an unique expression by which the line pattern detection could be realized.[1)]

$$\rho = x\cdot\cos\theta + y\cdot\sin\theta \qquad (1)$$

In order to extend the functional expression of Hough transform, we have proposed the extended Hough transform functions **(EHT)** and its functional conditions to be satisfied by EHT.[2)] As EHT functions are defined so that the new parameter space corresponds homeomorphically to (ρ,θ) space, it is emphasized in this paper that specifing the functional expression of EHT is equivalent to specifing the sensitivity distribution of the parameter space of Hough transform. The **'Warp Model'** of Hough transform was introduced to describe clearly the sensitivity distribution of the parameter space.

The conditions to be satisfied by EHT functions and the Warp Model of the EHT are firstly introduced, and after proving these conditions to be feasible, a procedure to design the transform function is presented by means of the sensitivity measures defined by the Warp Model. A few examples of the specific Warp Models are presented to show the utility of the Warp Model.

2. EHT AND WARP MODEL

The essential property of Hough transform is to ensure an unique one-to-one mapping, a hoeomorphic mapping, between a line in the pattern space and its parameter pair in the parameter space. If we could find other homeomorphic mapping functions, the functional expression of Hough transform would be extended. This new extended expression of Hough transform is called 'Extended Hough Transform (EHT)' in this paper.

2.1 Conditions for EHT

Let the EHT function be denoted by eq.(2) in this paper. The parameters μ and ξ in eq.(2) are for the parameters for the arbitrary position and for the arbitrary angle(or orientation) of the straight line, respectively. Any transform functions given by eq.(3) could be an EHT function on condition that the following three conditions be satisfied.

$$g(\mu,\xi) = x.f_1(\mu,\xi) + y.f_2(\mu,\xi) \qquad (2)$$

Condition-1: $f_1(\mu,\xi), f_2(\mu,\xi)$ and $g(\mu,\xi)$ should be an unique(single-valued) and continuous functions of the parameters μ and ξ, where $f_1^2+f_2^2 \neq 0$.

As $f_1^2+f_2^2 \neq 0$, eq.(2) can be replaced by eq.(3). In eq.(3), when let arctan $\frac{f_1(\mu,\xi)}{f_2(\mu,\xi)}$ and $\frac{g(\mu,\xi)}{\sqrt{f_1^2+f_2^2}}$ be replaced by $\phi(\mu,\xi)$ and $R(\mu,\xi)$, respectively, eq.(3) can simply be represented by eq.(4).

$$\frac{g}{\sqrt{f_1^2+f_2^2}} = x.\frac{f_1}{\sqrt{f_1^2+f_2^2}} + y.\frac{f_2}{\sqrt{f_1^2+f_2^2}} \qquad (3)$$

$$R(\mu,\xi) = x.\cos\phi(\mu,\xi) + y.\sin\phi(\mu,\xi) \qquad (4)$$

As functions $f_1(\mu,\xi)$, $f_2(\mu,\xi)$ and $g(\mu,\xi)$ are single-valued, the modified parameters (ϕ,R) in eq.(4) become single-valued functions of μ, and ξ, and therefore we can call these parameters as **equivalent distance** R from the origin and **equivalent perpendicular angle** ϕ of the EHT defined by eq.(2).

Condition-2: The equivalent distance $R(\mu,\xi)$ and perpendicular angle $\phi(\mu,\xi)$ must be monotonously increasing(or decreasing) with respect to μ and ξ, respectively. Therefore, eq.(5) must be satisfied. **Figure**

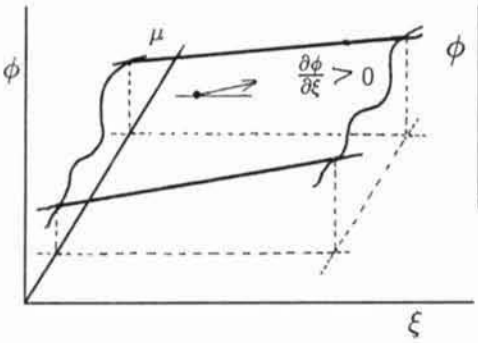

Fig.1 Condition-2 of EHT

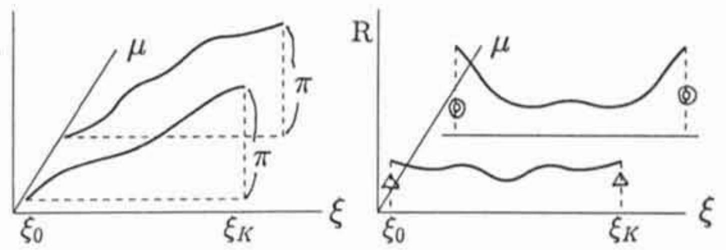

Fig.2 Condition-3 of EHT

1 gives the pictorial interpretation of this condition.

$$\frac{\partial R(\mu,\xi)}{\partial \mu} > (or <)0 \tag{5}$$

$$\frac{\partial \phi(\mu,\xi)}{\partial \xi} > (or <)0$$

Condition-3: When let the boudaries of the parameters ξ and μ be ξ_K, ξ_0, μ_L and μ_0, eq.(6) must be satisfied as the boundary conditions for the parameters ϕ and R for any μ and any ξ, where B is the given range of parameter ρ. **Figure 2** gives an interpretation of this condition.

$$|\phi(\mu_l,\xi_K) - \phi(\mu_l,\xi_0)| = \pi \tag{6}$$

$$R(\mu_l,\xi_K) - R(\mu_l,\xi_0) = 0$$

$$R(\mu_L,\xi_k) \geq \frac{B}{2} \quad R(\mu_0,\xi_k) \leq \frac{-B}{2}$$

2.2 Warp Model of EHT

In order to demonstrate the feasibilities of the conditions, suppose that the parameter spaces (ρ,θ) and (μ,ξ) were digitized into $L \times K$ cells homogeniously as shown in **Fig.3**. As the domain for the parameter space (ρ,θ) should be $0 \leq \theta < \pi$ and $\frac{-B}{2} \leq \rho < \frac{B}{2}$, and if let the domain of the space (μ,ξ) be $D \times C$, the relations between the cell sizes of $\triangle\rho \times \triangle\theta$ and $\triangle\mu \times \triangle\xi$ can be given by eq.(7). Let the borders of cells be identified by (ρ_l,θ_k) and (μ_l,ξ_k) where l=0,1,2,...,L and k=0,1,2,...,K.

$$\triangle\rho = \frac{B}{D}.\triangle\mu \tag{7}$$

$$\triangle\theta = \frac{\pi}{C}.\triangle\xi$$

How should the cell array (μ_l,ξ_k) in (μ,ξ) space be observed from an observer on the space (ρ,θ) ? As shown in **Fig.4**, it is obvious that the skewed cell array $(R(\mu_l,\xi_k),\phi(\mu_l,\xi_k))$ would be observed. We will call this skewed cell array as the 'Warp Model'[1] of EHT. The point to show that the EHT is truely an alternative of the Hough transform is to show the Warp Model could cover the cell array (ρ_l,θ_k) uniquely.

2.3 Proofs for Conditions

(1) Condition-1

[1] an analogy to the skewed warp and weft of the fabrics

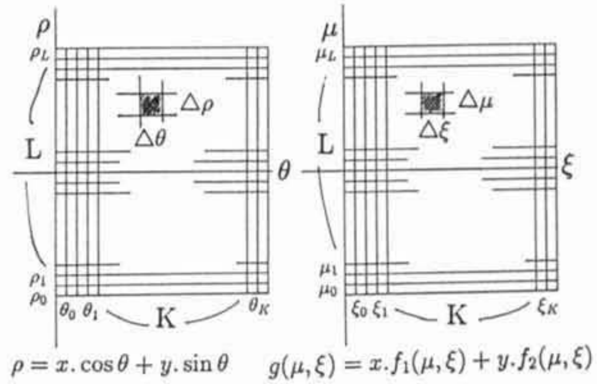

Fig.3 Relation between (μ,ξ) and (ρ,θ) spaces

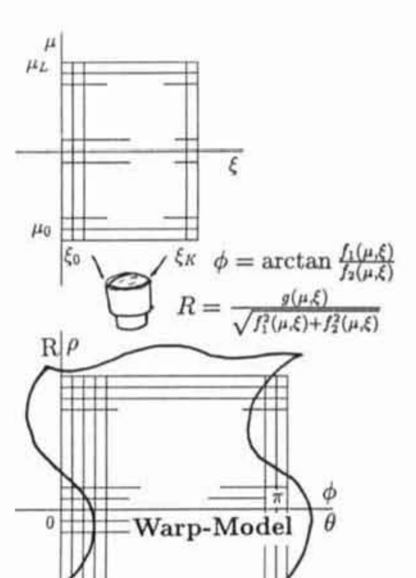

Fig.4 The Warp Model (R,ϕ) of EHT

In oder to ensure the correspondence between the cell array (ρ,θ) and $(R(\mu,\xi),\phi(\mu,\xi))$ to be homeomorphic, it is sufficient to ensure the functions $R(\mu,\xi)$ and $\phi(\mu,\xi)$ to be single-valued and continuous with respect to ρ and θ, respectively. Therefore, as known from eq's.(3) and (4), it is sufficient to ensure functions $f_1(\mu,\xi)$, $f_2(\mu,\xi)$ and $g(\mu,\xi)$ to be single-valued and countinuous with respect to μ and ξ, respectively.(**QED**)

(2) Condition-2
In addition to the Condition-1, as shown in **Fig.5**, it would be feasible that the functions $R(\mu,\xi)$ and $\phi(\mu,\xi)$ should be monotonously increasing (or decreasing) with respect to ρ and θ, respectively. Therefore, eq.(8) can be easily provided by means of eq.(7).(**QED**)

$$\frac{\partial\phi(\mu,\xi)}{\partial\theta} = \frac{C}{\pi}.\frac{\partial\phi(\mu,\xi)}{\xi} > (<)0 \quad (8)$$
$$\frac{\partial R(\mu,\xi)}{\partial\rho} = \frac{D}{B}.\frac{\partial R(\mu,\xi)}{\partial\mu} > (<)0$$

(3) Condition-3
As shown in **Fig.6**, in order to ensure the Warp Model $R(\mu_l,\xi_k)$ to cover the cell array (ρ,θ), eq.(9) must be satisfied for all μ_l, l=0,1,2,...,L.
In addition to eq.(9), for all ξ_k except k=0 and K, eq.(10) should be applicable where B is the given range for ρ parameter.(**QED**)

$$R(\mu_l,\xi_K) = R(\mu_l,\xi_0) \quad (9)$$
$$\phi(\mu_l,\xi_K) = \phi(\mu_l,\xi_0) + \pi$$
$$R(\mu_L,\xi_k) \geq \frac{B}{2} \quad R(\mu_0,\xi_k) \leq \frac{-B}{2} \quad (10)$$

3. HOW TO DESIGN EHT FUNCTION

A basic idea and a procedure to design the EHT function are proposed by means of the Warp Model.

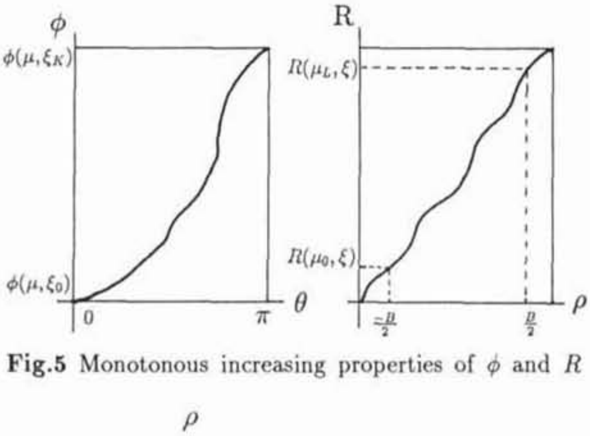

Fig.5 Monotonous increasing properties of ϕ and R

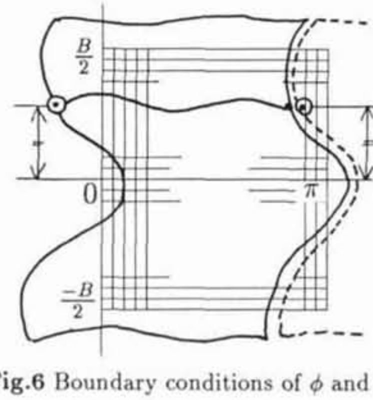

Fig.6 Boundary conditions of ϕ and R

3.1 Sensitivity of Line Detection

It is important to notice that the partial differential $\frac{\partial\phi}{\partial\theta}$ defined by eq.(8) can be physically interpreted as a kind of 'the angular sensitivity' or 'the angular resolution rate' of the given EHT characterized by $(R(\mu,\xi),\phi(\mu,\xi))$ in eq.(4).
In the same way, the partial differential $\frac{\partial R}{\partial\rho}$ means a kind of 'the positional sensitivity' of the line detection by the EHT. In other words, the skewness of the Warp Model of the EHT represents the angular and positional sensitivities of the EHT.
Let S_ϕ and S_R be the angular and positional sensitivities. Fortunately, as clearly given in eq.(8), these sensitivities can be defined by eq.(11).

$$S_\phi = \frac{C}{\pi}.\frac{\partial\phi}{\partial\xi} \quad (11)$$
$$S_R = \frac{B}{D}.\frac{\partial R}{\partial\mu}$$

3.2 Procedure to design EHT functions

Therefore, the exact expression of the EHT function can be derived by solving the differential equations given by S_ϕ and S_R in eq.(11) under the constraint of the Condition-1,2, and 3. The solutions $\phi(\mu,\xi)$ and $R(\mu,\xi)$ are the exact expression of the EHT in which the specified sensitivities are actually realized.

3.3 Examples of EHT

A few examples of EHT functions are presented to demonstrate the procedure.

(1) Example(1)
Suppose to design a EHT function of which angular and positional sensitivities are increasing at about $\theta = \frac{\pi}{2}$ as given by eq.(12) and **Fig.7**.

$$S_\phi = \frac{C}{\pi}.(\xi - \frac{\pi}{2})^2 + a \quad (12)$$
$$S_R = \frac{B}{D}.(\xi - \frac{\pi}{2})^2 + a'$$

The parameter $\phi(\mu,\xi)$ can be easily solved to be eq.(13) on condition that $C = \pi$ when $0 \leq \xi < \pi$, and that $a = (1 - \frac{\pi^2}{12})$ and $b = \frac{\pi^3}{24}$ when $\phi(\xi = 0) = 0$ and $\phi(\xi = \pi) = \pi$.
The another parameter $R(\mu,\xi)$ can also be derived by eq.(14) on condition that $a' = (1 - \frac{\pi^2}{12})$ and $b' = 0$ when $R(\mu = 0) = 0$, $R(\mu = R) = R$ and B=D.
Therefore, the transform function of this EHT was designed as $R(\mu,\xi) = x.\cos\phi(\mu,\xi) + y.\sin\phi(\mu,\xi)$ on the basis of the solusions of eq.'s (13) and (14). **Figure 8** shows the skewed parameter space, Warp Model, of this EHT.

$$\phi(\mu,\xi) = \int S_\phi d\xi = \frac{(\xi - \frac{\pi}{2})^3}{3} + a\xi + b$$
$$= \frac{(\xi - \frac{\pi}{2})^3}{3} + (1 - \frac{\pi^2}{12}).\xi + \frac{\pi^3}{24} \quad (13)$$

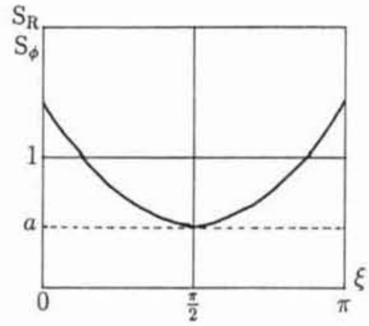

Fig.7 An example of the sensitivities $\frac{\partial \phi}{\partial \xi}$ and $\frac{\partial R}{\partial \mu}$

$$R(\mu,\xi) = \int S_R d\mu = \frac{B}{D} \cdot \int \{(\xi - \frac{\pi}{2})^2 + a'\} d\mu$$

$$= \mu.(\xi - \frac{\pi}{2})^2 + (1 - \frac{\pi^2}{12}) \quad (14)$$

(2) Example (2)
Suppose to introduce a EHT function such that its angular sensitivity becomes linear as defined by eq.(15) and that its positional sensitivity becomes flat as defined by eq.(16).
Under the same constraints of the example (1), $\phi(\mu,\xi)$ and $R(\mu,\xi)$ are easily solved, and eq.(17) was provided as the EHT transform function. **Figure 9** shows the Warp Model of this EHT.

$$S_\phi = \frac{2}{\pi}.\xi \quad (15)$$

$$S_R = 1 \quad (16)$$

$$R(\mu,\xi) = \mu \quad \phi(\mu,\xi) = \frac{\xi^2}{\pi} \quad (17)$$

$$R(\mu,\xi) = x.\cos\phi(\mu,\xi) + y.\sin\phi(\mu,\xi)$$

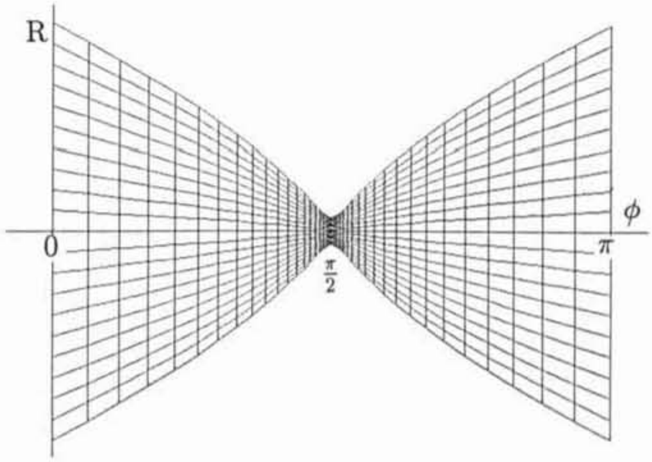

Fig.8 An example of the Warp Model

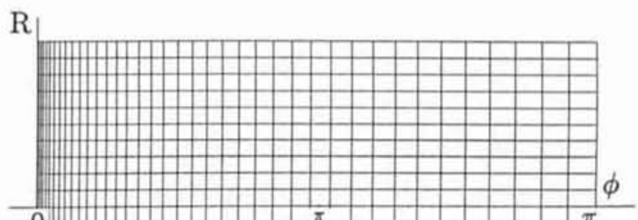

Fig.9 Another example of the Warp Model

4. CONCLUSION

This paper proposed the Warp Model of the Hough parameter space in order to give a pictorial interpretation of the extended Hough transform(EHT), and to introduce a procedure to design the EHT function. As the partial differentials of the Warp Model mean a kind of the sensitivity of the line extraction, just by specifing the sensitivity at request and by solving the partial differentials under the constraints, the new EHT function can be provided.
Therefore, it can be said that this paper should be effective to integrate the so-called 'performance' [6], 'quantization' [4),5)] and 'parameterization' problems of the Hough transform. From this view point, the following subjects of this paper should be investigated hereafter.
(1) To find out the fast, noise robust algorithms of EHT[3)]
(2) To make the functional conditions of EHT more target specific
(3) To clarify the relations among the transform function(=EHT), parameterization, and performance problems

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